MODIFICATION DESIGN WITH MINIMAL ROTARY INERTIA FOR THE PITCH CURVE WITH CONCAVE CUSPS OF *N*-LNG

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ABSTRACT: *To improve machining quality of teeth profile and transmission performance of N-lobed noncircular gear (N-LNG), the calculus of variations is employed to establish a modification model of design defect for the pitch curve with concave cusps that takes rotary inertia into consideration. A general minimal rotary inertia modification algorithm for pitch curve with concave cusps can be developed based on the model. The pitch curve without concave cusps for any type of N-LNG can be obtained by applying the proposed method. Outer contact and inner contact pitch curves conjugated with the modified pitch curve can also be obtained by using the principle of gearing. Several typical examples are implemented in computer, and simulation results demonstrate that the research should be helpful in the design and manufacture of N-LNG.*

KEYWORDS: *noncircular gear, pitch curve design, concave cusp defect, mathematical model, minimal rotary inertia modification*

1 INTRODUCTION

Noncircular gear pair has been used to implement variable transmission and function generation with applications in pump, printing, packaging, automotive steering, aeronautical ring dampers, knee motion assist mechanism, power drive mechanism and etc (Litvin, 2004, 2008), (Dooner, 2001), (Stefano, 2012), (Terada, 2012), (Mundo, 2006). However, because of the complex of design and manufacture for noncircular gears with discretionary pitch curves, its application is not as well as circular gears widely. One of important reasons is most research on noncircular gears focused on *N*-lobed or high-order elliptical gears (Bair, 2002), (Tsay, 2005), (Tong, 1998). Figliolini G. and co-workers proposed a general generation method of *N*-lobed elliptical gears from a basic ellipse, and the transmission mathematical models among cutter, driving and driven elliptical gears can be established in their studies (Figliolini, 2000, 2003, 2005). Moreover, the synthesis of high-order elliptical gears and its hobbing machining method can be obtained (Lin, 2013), (Zhang, 2014), (Figliolini, 2016).

More recently, some novel pitch curves of noncircular gears can be studied so as to expand its application. For examples, Hector, Salvador and et al designed an approach for generating pitch curves of *N*-lobed noncircular gears based on Bézier and B-spline nonparametric curves (Hector, 2007). Yao presented a pitch curve design method, which using plane regular *N*-curved polygon and spiral of Archimedes as the pitch curve for *N*-lobed noncircular gear (Yao, 2013). The Archimedes spiral, quadratic curve and Pascal curve are applied in pitch surface design of noncircular bevel gear (Lv, 2016). Noncircular gears with steepest rotation pitch curves can be synthesized, and the research is applied to modify the pitch curve with discontinuity points (Zhang, 2016a, 2016b).

Because of the complex of manufacture for pitch curve with cusps, the pitch curve should be reshaped to improve the machining process. Therefore, the minimal rotary inertia modification model of pitch curve with concave cusps for *N*-LNG can be established by the kinematics and calculus of variations in this paper due to the large rotary inertia will increase mechanical load and reduce response of the transmission system. Besides, we develop a modification algorithm for the design of conjugate pitch curves without concave cusps for *N*-LNGs. Numerical results are shown to illustrate the proposed method should be helpful in the design and manufacture of noncircular gears. They should also be helpful in the low load and high response of this type of gearing.

2 MODIFICATION MODEL OF THE PITCH CURVE WITH CONCAVE CUSPS

As shown in Fig.1, points *A*, *B* and *O* are non-collinear points in the plane, using straight line connect the points *O* and *A*, and points *O* and *B*, respectively. Let the *X*-axis of fixed coordinate system $\Gamma(O\text{-}XY)$ coincide with the line *OA*. While using discretionary plane curve $r_1(\eta)$ connects points *A* and *B*, then $\angle AOB$ is center angle of curve $r_1(\eta)$, polar angle η is measured counterclockwise from the positive direction *X*-axis. In order to ensure that the closed of the noncircular gear pitch curve, curve $r_1(\eta)$ should satisfy the following conditions:

$$\begin{cases} \angle AOB = 2\pi / N \\ r_1(0) = r_1(2\pi / N) \end{cases} \quad (1)$$

where *N* is an integer number.

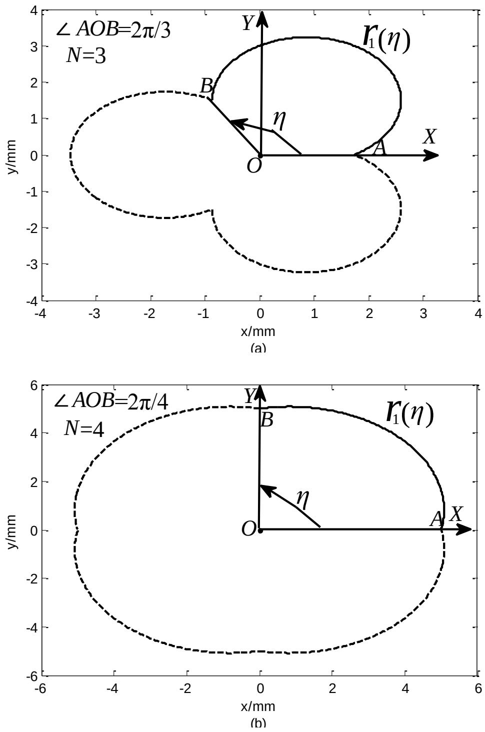

Fig. 1 Design of discretionary closed pitch curve for *N*-LNG

N-lobed closed plane curve $r(\eta)$ can be obtained by rotating the curve $r_1(\eta)$ around the center *O*, it can be expressed as:

$$r(\eta) = \begin{cases} r_1(\eta), \eta = [0, \dfrac{2\pi}{N}] \\ \vdots \\ r_k(\eta) = r_1(\eta - \dfrac{2(k-1)\pi}{N}), \quad \eta = [\dfrac{2(k-1)\pi}{N}, \dfrac{2k\pi}{N}] \end{cases} \quad (2)$$

where k=1,2,...,*N*.

Therefore, discretionary closed pitch curve $r(\eta)$ of *N*-LNG can be formed by rotating the plane curve $r_1(\eta)$ around the center *O*.

As shown in Fig.2, points c_1, c_2, ..., c_N may be concave cusps (i.e. continuous but non-differentiable points) of the pitch curve $r(\eta)$ for *N*-LNG. The design and machining of *N*-LNG have great difficulties due to the presence of these concave cusps. Modification of the pitch curve with concave cusps is necessary and important according to processing requirements of *N*-LNG and positions of these concave cusps have a certain rotated period 2π/*N*. For convenience, therefore, we just need to study the modification model of one concave cusp. Assuming the modification pitch curve corresponding concave cusp c_k (k=1,2,...,*N*) is $r_{Mk}(\eta)$, which used to replace two tiny section pitch curves (curves $\widehat{a_k c_k}$ and $\widehat{c_k b_k}$), a_k and b_k are points of tangency that the modification pitch curve $r_{Mk}(\eta)$ makes with pitch curve $r_{Nk}(\eta)$ and $r_{N(k+1)}(\eta)$, respectively. Without additional explanation, the following discussion relating to subscript k (k=1,2,...,*N*), if $k=N$, then $(k+1)\equiv 1$, and if $k=1$, then $(k-1)\equiv N$. Assuming the polar angles of points a_k and b_k are η_{ak} and η_{bk} respectively. So the infinitesimal arc dS of modification pitch curve $r_{Mk}(\eta)$ can be expressed as:

$$dS = \sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}\, d\eta \quad (3)$$

where subscript 'M*k*' in curve $r_{Mk}(\eta)$ is refer to that curve $r_{Mk}(\eta)$ is the modification pitch curve $r_{Mk}(\eta)$ for pitch curve $r_k(\eta)$ at the concave cusp c_k.

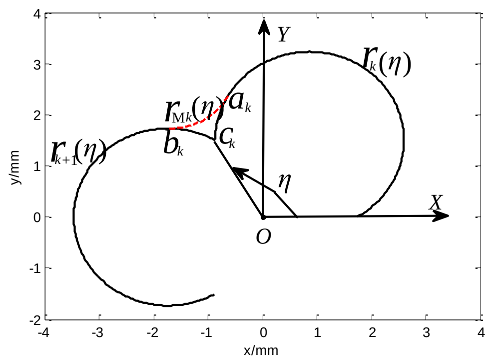

Fig. 2 Modification model of the pitch curve with concave cusps

According to the principle of kinematics and differential, assuming the mass of noncircular gear pitch curve is m, an infinitesimal dm can be chosen on the modification pitch curve $r_{Mk}(\eta)$, its mass is $dm=\lambda dS$, where parameter λ is linear density and the value is m/S, then the infinitesimal rotary inertia dJ can be expressed as:

$$dJ = r_{Mk}^2(\eta)\, dm = r_{Mk}^2(\eta) \frac{m}{S} dS \quad (4)$$

Substituting Eq.(3) into Eq.(4), the infinitesimal rotary inertia differential dJ can be obtained:

$$dJ = \frac{m r_{Mk}^2(\eta)\sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}}{\int_{\eta_{ak}}^{\eta_{bk}} \sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}\,d\eta} d\eta \quad (5)$$

The integral to both sides of Eq.(5) at the same time, one obtains:

$$J[r_{Mk}(\eta)] = \int_{\eta_{ak}}^{\eta_{bk}} \frac{m r_{Mk}^2(\eta)\sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}}{\int_{\eta_{ak}}^{\eta_{bk}} \sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}\,d\eta} d\eta \quad (6)$$

Referring to Eq.(6), we know that the rotary inertia J may be different for different modification pitch curves $r_{Mk}(\eta)$, the minimum value $J_{\min}$ of rotary inertia J can be expressed as:

$$\begin{cases} J_{\min}\left[r_{Mk}(\eta)\right] = \min\limits_{\substack{r_{Mk}(\eta)\in C^{(2)}[\eta_{ak},\eta_{bk}] \\ r_{Mk}(\eta_{ak})=r_k(\eta_{ak}),\, r_{Mk}(\eta_{bk})=r_{k+1}(\eta_{bk})}} \int_{\eta_{ak}}^{\eta_{bk}} F\,d\eta \\ F(r_{Mk}, r_{Mk}') = \dfrac{m r_{Mk}^2(\eta)\sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}}{\int_{\eta_{ak}}^{\eta_{bk}} \sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}\,d\eta} \end{cases} \quad (7)$$

where $r_{Mk}(\eta) \in C^2[\eta_{ak}, \eta_{bk}]$ refers to $r_{Mk}(\eta)$ have 2-order successive derived function at $\eta \in [\eta_{ak}, \eta_{bk}]$.

According to the calculus of variations (Lao, 2015) and generic function F of Eq. (7) doesn't contain parameter η, the first integral of Euler-Lagrange Equation can be expressed as:

$$F - r_{Mk}'(\eta) F_{r_{Mk}'(\eta)} = h \quad (8)$$

where h is an integral constant.

Substituting F of Eq.(7) into Eq.(8), we can obtain:

$$\frac{r_{Mk}^4(\eta)}{\sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}} = h_1 \quad (9)$$

where $h_1 = \dfrac{h\int_{\eta_{ak}}^{\eta_{bk}} \sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}\,d\eta}{m}$.

In order to avoid the occurrence of new concave cusps after the changes, the modified pitch curve $r_{Mk}(\eta)$ with minimal rotary inertia should meet the following differential conditions:

$$\begin{cases} r_{Mk}'(\eta_{ak}) = r_k'(\eta_{ak}) \\ r_{Mk}'(\eta_{bk}) = r_{k+1}'(\eta_{bk}) \end{cases} \quad (10)$$

Along with Eq.(9), modification model with minimal rotary inertia for the pitch curve with concave cusps of N-LNG can be obtained:

$$\begin{cases} \dfrac{r_{Mk}^4(\eta)}{\sqrt{r_{Mk}^2(\eta) + r_{Mk}'^2(\eta)}} = h_1 \\ s.t. \begin{cases} r_{Mk}(\eta) \in C^{(2)}[\eta_{ak}, \eta_{bk}] \\ r_{Mk}(\eta_{ak}) = r_k(\eta_{ak}) \\ r_{Mk}'(\eta_{ak}) = r_k'(\eta_{ak}) \\ r_{Mk}(\eta_{bk}) = r_{k+1}(\eta_{bk}) \\ r_{Mk}'(\eta_{bk}) = r_{k+1}'(\eta_{bk}) \end{cases} \end{cases} \quad (11)$$

where *s.t.* is an abbreviation of "subject to", it means that contained constraint conditions must be satisfied by the equation.

3 ANALYSIS AND SOLUTION OF THE MODIFICATION MODEL

The solution of Eq.(11) is called the modification pitch curve $r_{Mk}(\eta)$ with minimal rotary inertia corresponding concave cusp c_k. Referring to Eq.(11), assuming:

$$r_{Mk}'(\eta) = r_{Mk}(\eta)\tan\mu \quad (12)$$

Substituting Eq.(12) into Eq.(11), we can obtain:

$$r_{Mk}^3(\eta)\cos\mu = h_1 \quad (13)$$

$$dr_{Mk}(\eta) = \frac{r_{Mk}(\eta)\tan\mu}{3} d\mu = \frac{r_{Mk}'(\eta)}{3} d\mu \quad (14)$$

$$d\eta = \frac{dr_{Mk}(\eta)}{r_{Mk}'(\eta)} = \frac{d\mu}{3} \quad (15)$$

The integral to both sides of Eq.(15) at the same time, one obtains:

$$3\eta = \mu + h_2 \quad (16)$$

where h_2 is an integral constant.

Along with Eq.(13), the modification pitch curve $r_{Mk}(\eta)$ with minimal rotary inertia can be expressed as:

$$r_{Mk}(\eta) = \sqrt[3]{\frac{h_1}{\cos(3\eta - h_2)}}, \eta \in [\eta_{ak}, \eta_{bk}] \quad (17)$$

where h_1, h_2 are undetermined constants, polar angles η_{ak} and η_{bk} must satisfy the following constraints due to the value of pitch curve $r_{Mk}(\eta)$ is constant greater than zero:

$$\eta \in [\eta_{ak}, \eta_{bk}] \subset \left(\frac{h_2}{3} + \frac{\pi}{6} + \frac{2i\pi}{3}, \frac{h_2}{3} + \frac{\pi}{2} + \frac{2i\pi}{3}\right), h_1 < 0 \quad (18)$$

$$\eta \in [\eta_{ak}, \eta_{bk}] \subset \left(\frac{h_2}{3} - \frac{\pi}{6} + \frac{2i\pi}{3}, \frac{h_2}{3} + \frac{\pi}{6} + \frac{2i\pi}{3}\right), h_1 > 0 \quad (19)$$

where i=...,-2,-1,0,1,2,....

According to Eq.(12) and Eq.(16), the first-order derivative of modification pitch curve $r_{Mk}(\eta)$ can be obtained:

$$r'_{Mk}(\eta)=\tan(3\eta-h_2)\sqrt[3]{\frac{h_1}{\cos(3\eta-h_2)}},\eta\in[\eta_{ak},\eta_{bk}] \quad (20)$$

Along with Eqs.(17)~(19), we can obtain:

$$\begin{cases}\tan(3\eta-h_2)<0,\eta\in\left(\frac{h_2}{3}+\frac{\pi}{6}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{3}+\frac{2i\pi}{3}\right)\\ \tan(3\eta-h_2)>0,\eta\in\left(\frac{h_2}{3}+\frac{\pi}{3}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{2}+\frac{2i\pi}{3}\right)\end{cases},h_1<0 \quad (21)$$

$$\begin{cases}\tan(3\eta-h_2)<0,\eta\in\left(\frac{h_2}{3}-\frac{\pi}{6}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{2i\pi}{3}\right)\\ \tan(3\eta-h_2)>0,\eta\in\left(\frac{h_2}{3}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{6}+\frac{2i\pi}{3}\right)\end{cases},h_1>0 \quad (22)$$

$$\begin{cases}r'_{Mk}(\eta)<0,\eta\in\left(\frac{h_2}{3}+\frac{\pi}{6}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{3}+\frac{2i\pi}{3}\right)\\ r'_{Mk}(\eta)>0,\eta\in\left(\frac{h_2}{3}+\frac{\pi}{3}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{2}+\frac{2i\pi}{3}\right)\end{cases},h_1<0 \quad (23)$$

$$\begin{cases}r'_{Mk}(\eta)<0,\eta\in\left(\frac{h_2}{3}-\frac{\pi}{6}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{2i\pi}{3}\right)\\ r'_{Mk}(\eta)>0,\eta\in\left(\frac{h_2}{3}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{6}+\frac{2i\pi}{3}\right)\end{cases},h_1>0 \quad (24)$$

According to boundary conditions $r_{Mk}(\eta_{ak})=r_k(\eta_{ak})$, $r_{Mk}(\eta_{bk})=r_{k+1}(\eta_{bk})$ and differential conditions $r'_{Mk}(\eta_{ak})=r'_k(\eta_{ak})$, $r'_{Mk}(\eta_{bk})=r'_{k+1}(\eta_{bk})$, undetermined constants h_1, h_2 and polar angles η_{ak}, η_{bk} can be obtained by the following conditions:

$$\begin{cases}\sqrt[3]{\frac{h_1}{\cos(3\eta_{ak}-h_2)}}=r_k(\eta_{ak})\\ \sqrt[3]{\frac{h_1}{\cos(3\eta_{bk}-h_2)}}=r_{k+1}(\eta_{bk})\\ \tan(3\eta_{ak}-h_2)\sqrt[3]{\frac{h_1}{\cos(3\eta_{ak}-h_2)}}=r'_k(\eta_{ak})\\ \tan(3\eta_{bk}-h_2)\sqrt[3]{\frac{h_1}{\cos(3\eta_{bk}-h_2)}}=r'_{k+1}(\eta_{bk})\end{cases} \quad (25)$$

Referring to Fig.2, we known that the first-order derivatives $r'_k(\eta_{ak})$ and $r'_{k+1}(\eta_{bk})$ around the concave cusp c_k should satisfy the following conditions:

$$\begin{cases}r'_k(\eta_{ak})<0\\ r'_{k+1}(\eta_{bk})>0\end{cases} \quad (26)$$

According to Eqs.(17)~(26), the polar equation of the modification pitch curve $r_{Mk}(\eta)$ with minimal rotary inertia can be expressed as:

$$\begin{cases}r_{Mk}(\eta)=\sqrt[3]{\frac{h_1}{\cos(3\eta-h_2)}},\eta\in[\eta_{ak},\eta_{bk}]\\ h_1<0\\ \eta_{ak}\in\left(\frac{h_2}{3}+\frac{\pi}{6}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{3}+\frac{2i\pi}{3}\right)\\ \eta_{bk}\in\left(\frac{h_2}{3}+\frac{\pi}{3}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{2}+\frac{2i\pi}{3}\right)\end{cases} \quad (27)$$

$$\begin{cases}r_{Mk}(\eta)=\sqrt[3]{\frac{h_1}{\cos(3\eta-h_2)}},\eta\in[\eta_{ak},\eta_{bk}]\\ h_1>0\\ \eta_{ak}\in\left(\frac{h_2}{3}-\frac{\pi}{6}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{2i\pi}{3}\right)\\ \eta_{bk}\in\left(\frac{h_2}{3}+\frac{2i\pi}{3},\frac{h_2}{3}+\frac{\pi}{6}+\frac{2i\pi}{3}\right)\end{cases} \quad (28)$$

Along with Eq.(2), the sketch of modified pitch curve $r_M(\eta)$ is shown in Fig.3, and the polar equation $r_M(\eta)$ can be expressed as:

$$r_M(\eta)=\begin{cases}\begin{cases}r_1(\eta),\eta\in[\eta_{bN},\eta_{a1}]\\ r_{M1}(\eta)=\sqrt[3]{\frac{h_1}{\cos(3\eta-h_2)}},\eta\in[\eta_{a1},\eta_{b1}]\end{cases},k=1\\ \begin{cases}r_k(\eta)=r_1\left(\eta-\frac{2(k-1)}{N}\pi\right),\eta\in[\eta_{b(k-1)},\eta_{ak}]\\ r_{Mk}(\eta)=r_{M1}\left(\eta-\frac{2(k-1)}{N}\pi\right),\eta\in[\eta_{ak},\eta_{bk}]\end{cases},k=2,3,\ldots N\end{cases} \quad (29)$$

where subscript 'M' in $r_M(\eta)$ is refer to that the curve $r_M(\eta)$ is the modified pitch curve $r_M(\eta)$ for pitch curve $r(\eta)$ with concave cusps, polar angles η_{at}, η_{bt} (t=1,2,...,N and $t\neq k$) should satisfy the following conditions:

$$\begin{cases}\eta_{at}=\eta_{ak}+\frac{2(t-k)\pi}{N}\\ \eta_{bt}=\eta_{bk}+\frac{2(t-k)\pi}{N}\end{cases} \quad (30)$$

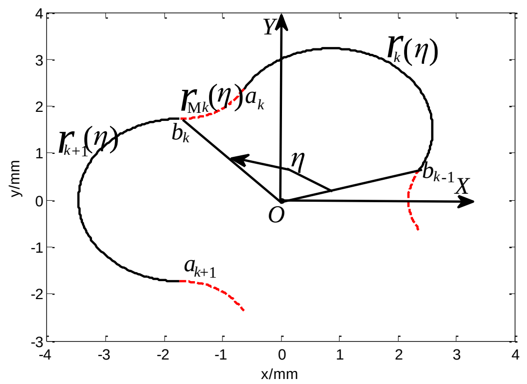

Fig. 3 Sketch of the modified pitch curve $r_M(\eta)$

Furthermore, the mathematical model of pitch curves conjugated with the modified pitch curve $r_M(\eta)$ which contain outer contact and inner contact can be expressed, respectively:

$$\begin{cases} r_{OM}(\eta_O) = a_O - r_M(\eta) \\ \eta_O = \int_0^{\eta} \dfrac{r_M(\eta)}{a_O - r_M(\eta)} d\eta \end{cases} \quad (31)$$

$$\begin{cases} r_{IM}(\eta_I) = a_I + r_M(\eta) \\ \eta_I = \int_0^{\eta} \dfrac{r_M(\eta)}{a_I + r_M(\eta)} d\eta \end{cases} \quad (32)$$

where $r_{OM}(\eta_O)$, $r_{IM}(\eta_I)$ and a_O, a_I are the pitch curves and center distances of outer contact and inner contact noncircular gears conjugated with the modified pitch curve $r_M(\eta)$, respectively.

In order to ensure the closure of pitch curves $r_{OM}(\eta_O)$ and $r_{IM}(\eta_I)$ conjugated with the modified pitch curve $r_M(\eta)$, the center distances a_O and a_I can be adjusted by the following conditions:

$$\frac{2\pi}{N_O} = \int_0^{\frac{2\pi}{N}} \frac{r_M(\eta)}{a_O - r_M(\eta)} d\eta \quad (33)$$

$$\frac{2\pi}{N_I} = \int_0^{\frac{2\pi}{N}} \frac{r_M(\eta)}{a_I + r_M(\eta)} d\eta \quad (34)$$

where N_O and N_I are the numbers of lobes of pitch curves $r_{OM}(\eta_O)$ and $r_{IM}(\eta_I)$.

4 MODIFICATION ALGORITHM

Modification algorithm with minimal rotary inertia characteristic for the pitch curve with concave cusps of N-LNG was produced using the foregoing formulations. Referring to the algorithm flow chart of Fig. 4, the pitch curve $r(\eta)$ with concave cusps for N-LNG is the known condition, concave cusp c_k can be chosen by designers, arbitrarily. Polar angles η_{ak}, η_{bk} and undetermined constants h_1, h_2 can be solved by resorting to Eq.(25), Eq.(27) or Eq.(25), Eq.(28). The modified pitch curve $r_M(\eta)$ can be obtained through Eqs.(27)~(30), its conjugate pitch curves $r_{OM}(\eta_O)$ and $r_{IM}(\eta_I)$ can also be obtained by Eqs.(31)~(34).

5 MODIFICATION EXAMPLES

The proposed modification method with minimal rotary inertia characteristic was implemented in computer to run several typical examples, as shown in Figs. 5~8. For convenience, the value of subscript k for selected concave cusp c_k is 1, without additional explanation, the following examples relating to subscript k of selected concave cusp c_k whose value is 1.

Fig. 5 is the 3-lobed noncircular gear pitch curve with concave cusps, its polar equation $r(\eta)$ can be expressed as:

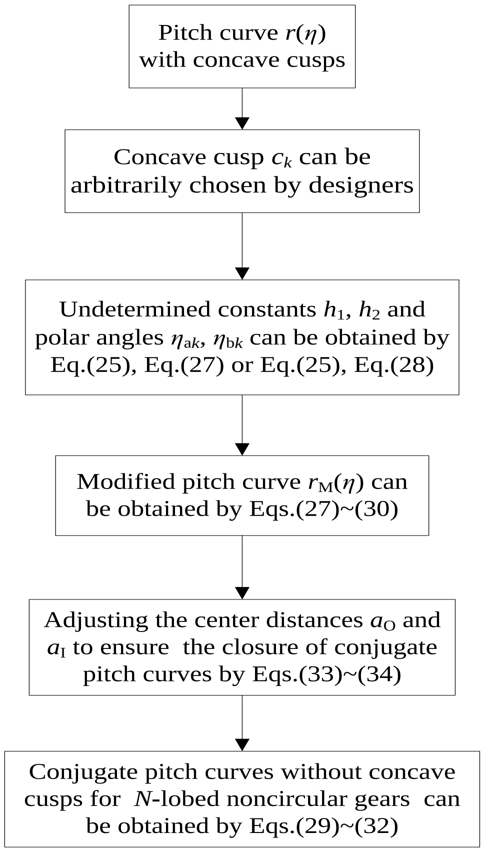

Fig. 4 Flow chart of the modification algorithm with minimal rotary inertia characteristic for pitch curve with concave cusps

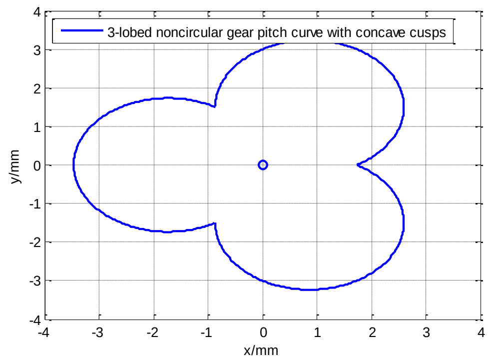

Fig. 5 3-lobed noncircular gear pitch curve with concave cusps

$$r(\eta) = \begin{cases} r_1(\eta) = 3\sin(\eta) + \sqrt{3}\cos(\eta), \eta \in [0, \dfrac{2\pi}{3}] \\ r_2(\eta) = r_1(\eta - \dfrac{2\pi}{3}), \eta \in [\dfrac{2\pi}{3}, \dfrac{4\pi}{3}] \\ r_3(\eta) = r_1(\eta - \dfrac{4\pi}{3}), \eta \in [\dfrac{4\pi}{3}, 2\pi] \end{cases} \quad (35)$$

Along with Eq.(25) and Eq.(27) or Eq.(25) and Eq.(28), undetermined constants h_1, h_2 and polar angles η_{a1}, η_{b1} of modification pitch curve $r_{M1}(\eta)$ for the pitch curve $r(\eta)$ with concave cusps in Fig. 5 can be obtained, respectively:

$$\begin{cases} h_1 = -6\sqrt{3} \\ h_2 = \pi - 2i\pi \\ \eta_{a1} = 7\pi/12 \\ \eta_{b1} = 3\pi/4 \end{cases} \quad (36)$$

$$\begin{cases} h_1 = 6\sqrt{3} \\ h_2 = 2\pi - 2i\pi \\ \eta_{a1} = 7\pi/12 \\ \eta_{b1} = 3\pi/4 \end{cases} \quad (37)$$

Substituting Eq.(36) or Eq.(37) into Eq.(27) that takes the periodic characteristic of trigonometric function into consideration, the polar equation of the modification pitch curve $r_{M1}(\eta)$ with minimal rotary inertia can be obtained:

$$r_{M1}(\eta) = \sqrt[3]{\frac{6\sqrt{3}}{\cos(3\eta)}}, \eta \in \left[\frac{7\pi}{12}, \frac{3\pi}{4}\right] \quad (38)$$

Then the modified pitch curves $r_M(\eta)$ for the pitch curve $r(\eta)$ with concave cusps in Fig. 5 can be obtained:

$$r_M(\eta) = \begin{cases} r_1(\eta) = 3\sin(\eta) + \sqrt{3}\cos(\eta), \eta \in [\frac{\pi}{12}, \frac{7\pi}{12}] \\ r_{M1}(\eta) = \sqrt[3]{\frac{6\sqrt{3}}{\cos(3\eta)}}, \eta \in [\frac{7\pi}{12}, \frac{3\pi}{4}] \\ r_2(\eta) = r_1(\eta - \frac{2\pi}{3}), \eta \in [\frac{3\pi}{4}, \frac{5\pi}{4}] \\ r_{M2}(\eta) = r_{M1}(\eta - \frac{2\pi}{3}), \eta \in [\frac{5\pi}{4}, \frac{17\pi}{12}] \\ r_3(\eta) = r_1(\eta - \frac{4\pi}{3}), \eta \in [\frac{17\pi}{12}, \frac{23\pi}{12}] \\ r_{M3}(\eta) = r_{M1}(\eta - \frac{4\pi}{3}), \eta \in [\frac{23\pi}{12}, \frac{25\pi}{12}] \end{cases} \quad (39)$$

Fig. 6(a) is the modified pitch curves $r_M(\eta)$ with the minimal rotary inertia for the pitch curve $r(\eta)$ with concave cusps in Fig. 5, its conjugate pitch curves equations and graphs which contain outer contact and inner contact are described in Eqs. (40)~(41) and Figs. 6(b)~(c).

$$\begin{cases} r_{OM}(\eta_O) = 5.9429 - r_M(\eta) \\ \eta_O = \int_0^{\eta} \frac{r_M(\eta)}{5.9429 - r_M(\eta)} d\eta \end{cases} \quad (40)$$

$$\begin{cases} r_{IM}(\eta_I) = 2.8686 + r_M(\eta) \\ \eta_I = \int_0^{\eta} \frac{r_M(\eta)}{2.8686 + r_M(\eta)} d\eta \end{cases} \quad (41)$$

Fig. 7 is the 4-lobed noncircular gear pitch curve with concave cusps, its polar equation $r(\eta)$ can be expressed as:

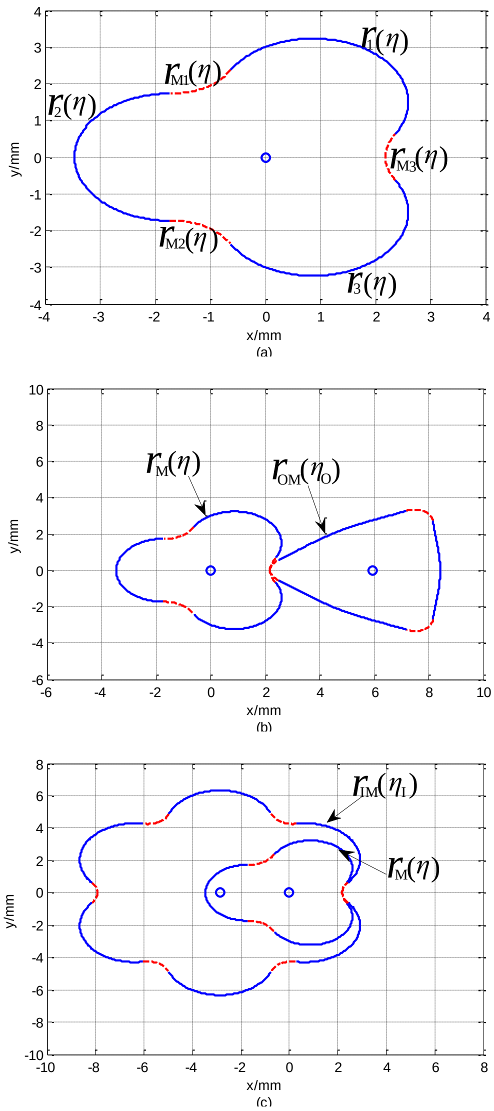

Fig. 6 Modified pitch curve rM(η) and its conjugate pitch curves for the pitch curve r(η) with concave cusps in Fig. 5

$$r(\eta) = \begin{cases} r_1(\eta) = \sin(\eta) + \cos(\eta), \eta \in [0, \frac{\pi}{2}] \\ r_2(\eta) = r_1(\eta - \frac{\pi}{2}), \eta \in [\frac{\pi}{2}, \pi] \\ r_3(\eta) = r_1(\eta - \pi), \eta \in [\pi, \frac{3\pi}{2}] \\ r_3(\eta) = r_1(\eta - \frac{3\pi}{2}), \eta \in [\frac{3\pi}{2}, 2\pi] \end{cases} \quad (42)$$

Referring to the modification process of 3-lobed noncircular gear pitch curve with concave cusps, the modified pitch curves $r_M(\eta)$ with the minimal rotary inertia for the pitch curve $r(\eta)$ with concave cusps in Fig. 7 is depicted in Fig. 8(a), its conjugate pitch curves which contain outer contact and inner contact are described in Figs. 8(b)~(c).

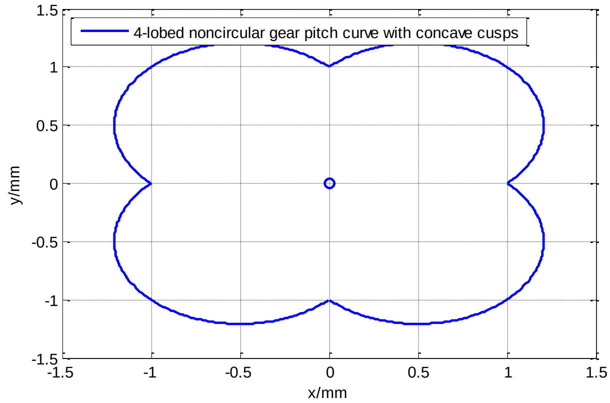

Fig. 7 4-lobed noncircular gear pitch curve with concave cusps

Corresponding modification parameters of the pitch curve with concave cusps in Fig. 7 are listed in Table 1, and the polar equations of modified pitch curve $r_M(\eta)$ can be expressed as:

$$r_M(\eta)=\begin{cases} r_1(\eta)=\sin(\eta)+\cos(\eta),\eta\in[\frac{\pi}{16},\frac{7\pi}{16}] \\ r_{M1}(\eta)=\sqrt[3]{\dfrac{2\sqrt{2}\cos^4(\frac{3\pi}{16})}{\cos(3\eta-\frac{3\pi}{2})}},\eta\in[\frac{7\pi}{16},\frac{9\pi}{16}] \\ r_2(\eta)=r_1(\eta-\frac{\pi}{2}),\eta\in[\frac{9\pi}{16},\frac{15\pi}{16}] \\ r_{M2}(\eta)=r_{M1}(\eta-\frac{\pi}{2}),\eta\in[\frac{15\pi}{16},\frac{17\pi}{16}] \\ r_3(\eta)=r_1(\eta-\pi),\eta\in[\frac{17\pi}{16},\frac{23\pi}{16}] \\ r_{M3}(\eta)=r_{M1}(\eta-\pi),\eta\in[\frac{23\pi}{16},\frac{25\pi}{16}] \\ r_4(\eta)=r_1(\eta-\frac{3\pi}{2}),\eta\in[\frac{25\pi}{16},\frac{31\pi}{16}] \\ r_{M4}(\eta)=r_{M1}(\eta-\frac{3\pi}{2}),\eta\in[\frac{31\pi}{16},\frac{33\pi}{16}] \end{cases} \quad (43)$$

Referring to Figs. 5-8, we known that the modification accuracy is higher with the decrease of the $\Delta\eta$ ($\Delta\eta=\eta_{b1}-\eta_{a1}$). Therefore, designer can choose appropriate polar angles η_{a1} and η_{b1} to ensure the requirement of engineering accuracy.

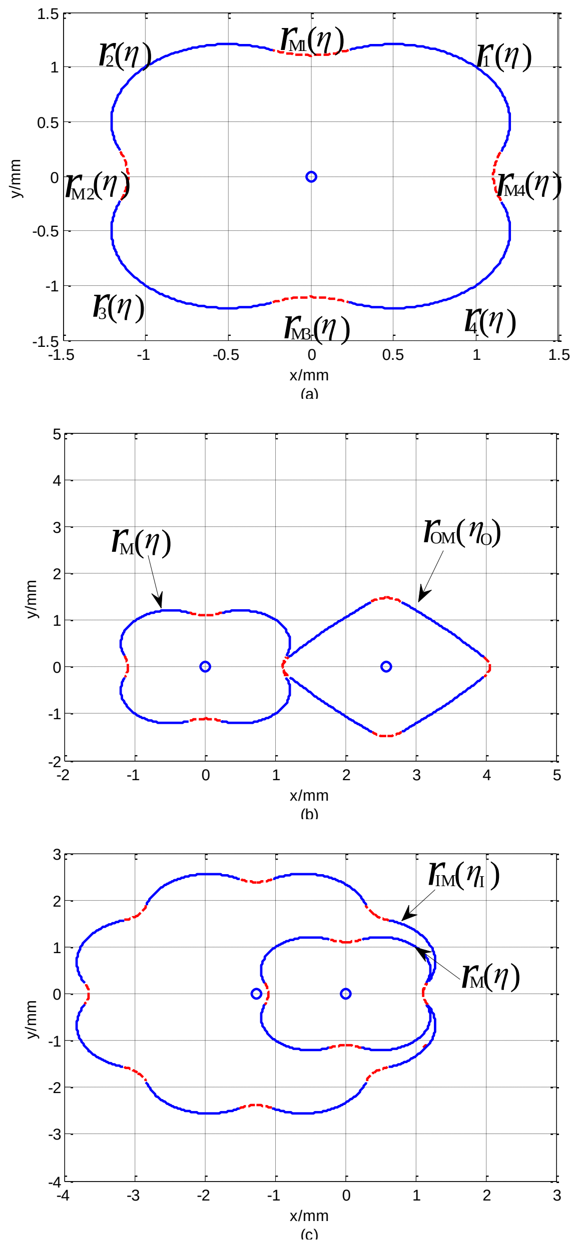

Fig. 8 Modified pitch curve rM(η) and its conjugate pitch curves for the pitch curve r(η) with concave cusps in Fig. 7

Table 1. Modification parameters of the pitch curve $r(\eta)$ with concave cusps in Fig. 7

Polar angles η_{a1}, η_{b1}	Integral constants h_1, h_2	Modified pith curve $r_{M1}(\eta)$	Conjugate pitch curves	
			Outer contact	Inner contact
$\begin{cases}\eta_{a1}=\frac{7\pi}{16}\\ \eta_{b1}=\frac{9\pi}{16}\end{cases}$	$\begin{cases}h_1=-2\sqrt{2}\cos^4(\frac{3\pi}{16})\\ h_2=\frac{\pi}{2}-2i\pi\end{cases}$ or $\begin{cases}h_1=2\sqrt{2}\cos^4(\frac{3\pi}{16})\\ h_2=\frac{3\pi}{2}-2i\pi\end{cases}$	$\begin{cases}r_{M1}(\eta)=\sqrt[3]{\dfrac{2\sqrt{2}\cos^4(\frac{3\pi}{16})}{\cos(3\eta-\frac{3\pi}{2})}}\\ \eta\in\left[\frac{7\pi}{16},\frac{9\pi}{16}\right]\end{cases}$	$\begin{cases}r_{OM}(\eta_O)=2.5827-r_M(\eta)\\ \eta_O=\int_0^{\eta}\frac{r_M(\eta)}{2.5827-r_M(\eta)}d\eta\end{cases}$ (Referring to Fig. 8(b))	$\begin{cases}r_{IM}(\eta_I)=1.2778+r_M(\eta)\\ \eta_I=\int_0^{\eta}\frac{r_M(\eta)}{1.2778+r_M(\eta)}d\eta\end{cases}$ (Referring to Fig. 8(c))

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7 NOTATION

The following symbols are used in this paper:

a_k, b_k= points of tangency;

a_I, a_O= center distances of the pitch curves $r_{IM}(\eta_I)$ and $r_{OM}(\eta_O)$;

c_k= concave cusps;

dm= infinitesimal mass;

dJ= infinitesimal rotary inertia;

dS= infinitesimal arc length;

h_1, h_2= undetermined constants;

k= 1, 2, …, N;

m= mass of noncircular gear pitch curve;

$r(\eta)$= pitch curve equation of N-LNG;

$r_M(\eta)$= modified pitch curve equation;

$r_{Mk}(\eta)$= modification pitch curve equation;

$r_{IM}(\eta_I)$, $r_{OM}(\eta_O)$= pitch curve equations of outer contact and inner contact noncircular gears conjugated with the modified pitch curve $r_M(\eta)$;

N= number of lobes;

N_O, N_I= numbers of lobes of pitch curves $r_{OM}(\eta_O)$ and $r_{IM}(\eta_I)$;

J= rotary inertia;

J_{min}=minimum value of rotary inertia J;

S= arc length of the modification pitch curve $r_{Mk}(\eta)$;

η_{ak}, η_{bk}= Corresponding polar angles of points a and b;

λ= linear density;

$\Gamma(O\text{-}XY)$= fixed coordinate system.